Ukrainian Mathematical Journal, Vol. 52, No. 7, 2000

BRIEF COMMUNICATIONS

ON THE APPROXIMATION OF FUNCTIONS OF THE HÖLDER CLASS BY BIHARMONIC POISSON INTEGRALS

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UDC 517.5

We determine the exact value of the upper bound of the deviation of biharmonic Poisson integrals from functions of the Hölder class.

Let $L_{2\pi}^1$ denote the class of 2π-periodic integrable functions with the norm

$$\|f\|_1 = \int_{-\pi}^{\pi} |f(x)|dx$$

and let $C_{2\pi} = L_{2\pi}^{\infty}$ be the class of 2π-periodic continuous functions with the norm

$$\|f\|_\infty = \max_x |f(x)|.$$

The set of functions $f \in L_{2\pi}^p$, $p = 1, \infty$, satisfying the inequality

$$\|f(x+h) - f(x)\|_p \le |h| \tag{1}$$

is denoted by H_p^1 and is called the Hölder class.

The set of functions $f \in L_{2\pi}^p$, $p = 1, \infty$, satisfying the inequality

$$\|f(t+h) - 2f(t) + f(t-h)\|_p \le 2|h|$$

is denoted by H_p^2 and is called the class of quasismooth functions [1].

Consider the biharmonic Poisson integral

$$A_2(r, \theta) = \int_{-\pi}^{\pi} f(t+\theta) P_2(r, t)\, dt,$$

where

$$P_2(r,t) = \frac{(1-r^2)^2(1-r\cos t)}{2\pi(1-2r\cos t + r^2)^2}, \qquad 0 \le r < 1,$$

is the biharmonic Poisson kernel [2, 3].

Volyn University, Lutsk. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 52, No. 7, pp. 971–974, July, 2000. Original article submitted March 10, 1998; revision submitted February 25, 2000.

0041–5995/00/5207–1113 $25.00

Let

$$\mathcal{E}\left(H_p^\nu, r\right)_p = \sup_{f \in H_p^\nu} \| A_2(r,\theta) - f(\theta) \|_p, \quad p = 1, \infty, \quad \nu = 1, 2, \tag{2}$$

denote the upper bound of the deviations of functions of the class H_p^ν from their biharmonic Poisson integrals.

If a function $\varphi(1-r) = \varphi\left(H_p^\nu; A_2(r,\theta); (1-r)\right)$ such that

$$\mathcal{E}\left(H_p^\nu, r\right)_p = \varphi(1-r) + o\left(\varphi(1-r)\right)$$

as $r \to (1-0)$ is explicitly determined, then one says [4] that the Kolmogorov–Nikol'skii problem is solved for the operator $A_2(r,\theta)$ and class H_p^ν, $p = 1, \infty$, $\nu = 1, 2$.

In 1963, Kaniev [5] established the following asymptotic equality for $\mathcal{E}\left(H_\infty^2, r\right)_\infty$ as $r \to (1-0)$:

$$\mathcal{E}\left(H_\infty^2, r\right)_\infty = \frac{2}{\pi}(1-r) + \frac{\varepsilon_r}{\pi}, \quad \varepsilon_r = o(1-r). \tag{3}$$

In 1968, Pych [6] established the following asymptotic equality:

$$\mathcal{E}\left(H_\infty^2, r\right)_\infty = \frac{2}{\pi}(1-r) + O\left((1-r)^2 \ln \frac{1}{1-r}\right), \quad r \to 1-0. \tag{4}$$

Estimates (3) and (4) enable one to establish the first asymptotic constant (the Kolmogorov–Nikol'skii constant) [7] for the approximation of functions from the class H_∞^2 by their biharmonic Poisson integrals.

In the present paper, we determine the exact value of the upper bound of the deviations of biharmonic Poisson integrals from the functions of the Hölder class H_p^1, $p = 1, \infty$.

The following statement is true:

Theorem 1. *The quantity* $\mathcal{E}\left(H_p^1, r\right)_p$ *defined by equality (2) can be represented in the form of the following asymptotic series as* $r \to 1-0$:

$$\mathcal{E}\left(H_p^1, r\right)_p = \frac{2}{\pi}(1-r) + \frac{2}{\pi}(1-r)^2 \ln\frac{1}{1-r} + \frac{2\ln 2 + 1}{\pi}(1-r)^2 + \frac{2}{\pi}\sum_{k=3}^{\infty}\left\{\frac{1}{k}(1-r)^k \ln\frac{1}{1-r} + \gamma_k (1-r)^k\right\}, \tag{5}$$

where $p = 1, \infty$ *and*

$$\gamma_k = \frac{1}{k}\left(\ln 2 + \frac{1}{k} - \sum_{j=1}^{k-1}\frac{2^{-j}}{j}\right) - \frac{1}{(k-2)(k-1)2^{k-1}}.$$

Proof. First, we prove equality (5) for $p = \infty$.

Since

$$\int_{-\pi}^{\pi} P_2(r,t)\,dt = 1,$$

we have

$$A_2(r,\theta) - f(\theta) = \int_{-\pi}^{\pi} \{ f(t+\theta) - f(\theta)\} P_2(r,t)\,dt.$$

Hence, by using (1), we obtain the estimate

$$\left| A_2(r,\theta) - f(\theta) \right| \le \int_{-\pi}^{\pi} |t| P_2(r,t)\,dt.$$

Since the class H^1_∞ contains a function equal to $|t|$ on the segment $[-\pi, \pi]$ for which the last inequality turns into the equality, according to (2) we get

$$\mathcal{E}\left(H^1_\infty, r\right)_\infty = \int_{-\pi}^{\pi} |t| P_2(r,t)\,dt = 2\int_0^{\pi} t\, P_2(r,t)\,dt. \tag{6}$$

By using formulas 2.554(1) and 2.556 from [8], one can easily show that

$$\int \frac{1 - r\cos t}{\left(1 - 2r\cos t + r^2\right)^2}\,dt = \frac{1}{\left(1 - r^2\right)^2}\left[\frac{\left(r - r^3\right)\sin t}{1 - 2r\cos t + r^2} + \int \frac{1 - r^2}{1 - 2r\cos t + r^2}\,dt\right]$$

$$= \frac{1}{(1-r^2)^2}\left[\frac{(r - r^3)\sin t}{1 - 2r\cos t + r^2} + 2\arctan\left(\frac{1+r}{1-r}\tan\frac{t}{2}\right)\right].$$

Applying this equality to integral (6) and integrating by parts, we get

$$\mathcal{E}\left(H^1_\infty, r\right)_\infty = \pi - \frac{1 - r^2}{\pi}\int_0^{\pi} \frac{r\sin t}{1 - 2r\cos t + r^2}\,dt - \frac{2}{\pi}\int_0^{\pi} \arctan\left(\frac{1+r}{1-r}\tan\frac{t}{2}\right) dt$$

$$= -\frac{1 - r^2}{\pi}\int_0^{\pi} \frac{r\sin t}{1 - 2r\cos t + r^2}\,dt + \frac{1}{\pi}\int_0^{\pi} t\,\frac{1 - r^2}{1 - 2r\cos t + r^2}\,dt. \tag{7}$$

It is known that, for $0 \le r < 1$, the following identities are true:

$$\frac{1 - r^2}{1 - 2r\cos t + r^2} = 1 + 2\sum_{k=1}^{\infty} r^k \cos kt, \qquad \frac{r\sin t}{1 - 2r\cos t + r^2} = \sum_{k=1}^{\infty} r^k \sin kt.$$

Applying these identities to (7), we obtain

$$\mathcal{E}\left(H^1_\infty, r\right)_\infty = \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} r^k \int_0^{\pi} t\cos kt\,dt - \frac{1 - r^2}{\pi}\sum_{k=1}^{\infty} r^k \int_0^{\pi} \sin kt\,dt$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{r^{2k-1}}{(2k-1)^2} - \frac{2(1-r^2)}{\pi}\sum_{k=1}^{\infty} \frac{r^{2k-1}}{2k-1} = \frac{4}{\pi}\left[\frac{\pi^2}{8} - \sum_{k=0}^{\infty} \frac{r^{2k+1}}{(2k+1)^2}\right] - \frac{2(1-r^2)}{\pi}\sum_{k=1}^{\infty} \frac{r^{2k-1}}{2k-1}. \tag{8}$$

Further, we use the relation

$$\ln\frac{1+r}{1-r} = 2\sum_{k=1}^{\infty}\frac{r^{2k-1}}{2k-1}, \qquad |r|<1,$$

and the results of [9], according to which we get

$$\frac{\pi^2}{8} - \sum_{k=0}^{\infty}\frac{r^{2k+1}}{(2k+1)^2} = \frac{1}{2}\sum_{k=1}^{\infty}\left\{\frac{1}{k}(1-r)^k\ln\frac{1}{1-r} + \beta_k(1-r)^k\right\}, \qquad r\in(0,1),$$

where

$$\beta_k = \frac{1}{k}\left(\ln 2 + \frac{1}{k} - \sum_{j=1}^{k-1}\frac{2^{-j}}{j}\right) \tag{9}$$

for natural k. If r satisfies the indicated conditions, relation (8) turns into the asymptotic equality

$$\begin{aligned}\mathscr{E}\left(H^1_\infty, r\right)_\infty = \frac{2}{\pi}\Bigg[&(\ln 2+1)(1-r) + (1-r)\ln\frac{1}{1-r}\\ &+ \frac{(1-r)^2}{2}\ln\frac{1}{1-r} + \frac{\ln 2}{2}(1-r)^2 - \frac{1-r^2}{2}\ln\frac{1}{1-r} - \frac{1-r}{2}(1+r)\ln(1+r)\Bigg]\\ &+ \frac{2}{\pi}\sum_{k=3}^{\infty}\left\{\frac{1}{k}(1-r)^k\ln\frac{1}{1-r} + \beta_k(1-r)^k\right\}.\end{aligned} \tag{10}$$

To simplify the expression in square brackets in equality (10), we expand the function $g(r) = (1+r)\ln(1+r)$ in a Taylor series in powers of $(r-1)$. As a result, we obtain

$$\frac{2}{\pi}\left[(1-r)\ln 2 - \frac{1-r^2}{2}\ln(1+r)\right] = \frac{1}{\pi}\left[(1-r)^2(1+\ln 2) - \sum_{k=2}^{\infty}\frac{(1-r)^{k+1}}{k(k-1)2^{k-1}}\right].$$

Then

$$\begin{aligned}\mathscr{E}\left(H^1_\infty, r\right)_\infty = \frac{2}{\pi}\Bigg[&(1-r) + (1-r)\ln\frac{1}{1-r} + \frac{(1-r)^2}{2}\ln\frac{1}{1-r} - \frac{1-r^2}{2}\ln\frac{1}{1-r}\\ &+ \frac{1+2\ln 2}{2}(1-r)^2 - \sum_{k=2}^{\infty}\frac{(1-r)^{k+1}}{k(k-1)2^{k-1}}\Bigg] + \frac{2}{\pi}\sum_{k=3}^{\infty}\left\{\frac{1}{k}(1-r)^k\ln\frac{1}{1-r} + \beta_k(1-r)^k\right\}.\end{aligned} \tag{11}$$

Performing identity transformations, we get

$$\frac{1-r^2}{2}\ln\frac{1}{1-r} + \sum_{k=2}^{\infty}\frac{(1-r)^{k+1}}{k(k-1)2^{k-1}} = (1-r)\ln\frac{1}{1-r} - \frac{(1-r)^2}{2}\ln\frac{1}{1-r} + \sum_{k=3}^{\infty}\frac{(1-r)^k}{(k-1)(k-2)2^{k-2}}.$$

Taking into account relation (9) for the determination of the coefficients β_k for $k \geq 3$ and using (11), we obtain equality (5).

The proof of equality (5) for $p = 1$ follows from the known exact asymptotic equalities (established by Motornyi in [10]) between the upper bounds for the deviations of functions of the class H_1^1 from their biharmonic Poisson integrals in the metric $\|f\|_1$ and the corresponding upper bounds for the deviations of functions of the class H_∞^1 in the metric $\|f\|_\infty$. The theorem is proved.

Corollary 1. *Since [11]* $\mathscr{E}\big(H_p^1, r\big)_p = \mathscr{E}\big(H_p^2, r\big)_p$, $p = 1, \infty$, *the quantity* $\mathscr{E}\big(H_p^2, r\big)_p$ *can be represented in the form of the series on the right-hand side of equality (5).*

Corollary 2. *If all conditions of Theorem 1 are satisfied, then the following asymptotic equalities hold as* $r \to 1-0$:

$$\mathscr{E}\big(H_\infty^2, r\big)_\infty = \mathscr{E}\big(H_\infty^1, r\big)_\infty = \frac{2}{\pi}(1-r) + O\left((1-r)^2 \ln\frac{1}{1-r}\right), \tag{12}$$

$$\mathscr{E}\big(H_\infty^2, r\big)_\infty = \mathscr{E}\big(H_\infty^1, r\big)_\infty = \frac{2}{\pi}(1-r) + \frac{2}{\pi}(1-r)^2 \ln\frac{1}{1-r} + O\big((1-r)^2\big). \tag{13}$$

The asymptotic equalities (5) enable one to successively determine the Kolmogorov–Nikol'skii constants with any degree of accuracy. The asymptotic equality (12) is an improved version of equality (3) obtained by Kaniev, and the asymptotic equality (13) is an improved version of equality (4) obtained by Pych. It should be noted that, in the case of approximation of functions from the Hölder class H_p^1, $p = 1, \infty$, by Abel–Poisson singular integrals, a similar theorem that enables one to determine the Kolmogorov–Nikol'skii constants with any degree of accuracy was proved in [9].

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